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Introduction

- In this topic, we will
  - Review initial-value problems (IVPs)
  - Discuss the differences in approaches to finding or approximating solutions to IVPs
  - Introduce cubic splines
  - Describe the upcoming lectures

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Initial-value problems

- An initial-value problem (IVP) is can be:
  - The first derivative described in terms of the independent variable and the function

$$y^{(1)}(t) = f\left(t, y(t)\right) \qquad y^{(1)}(t) = ty(t) + t - 1$$

$$y(t_0) = y_0 \qquad y(0) = 1$$

  - The n^{th} derivative described in terms of the independent variable, lower derivatives and the function

$$y^{(n)}(t) = f\left(t, y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t)\right)$$

$$y(t_0) = y_0 \qquad y^{(3)}(t) = y^{(2)}(t) + 2y^{(1)}(t)\,y(t) + \sin(t)$$

$$y^{(1)}(t_0) = y_0^{(1)} \qquad y(1) = 2$$

$$\vdots \qquad y^{(1)}(1) = 3$$

$$y^{(n-1)}(t_0) = y_0^{(n-1)} \qquad y^{(2)}(1) = 4$$

Approximating solutions to initial-value problems

Initial-value problems

- A system of coupled IVPs, for example

$$y_1^{(1)}(t) = 0.02y_1(t) - 0.1y_1(t)\,y_2(t)$$

$$y_2^{(1)}(t) = -0.04y_2(t) + 0.02y_1(t)\,y_2(t)$$

$$y_1(0) = 5233$$

$$y_2(0) = 323$$

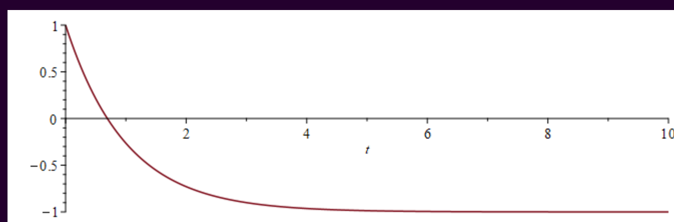
Solutions to IVPs

- Recall your approach in calculus:

$$y^{(1)}(t) = -y(t) - 1$$
$$y(0) = 1$$

- In calculus, you find a single exact solution:

$$y(t) = 2e^{-t} - 1$$



  – What if you cannot find an exact solution?

Approximate solutions to IVPs

- What do we have?

$$y^{(1)}(t) = -ty(t) - 1$$
$$y(0) = 1$$

  – At time $t = 0$, the value is 1
  – The first equation says:
    - If $t = 0$ and $y(0) = 1$, then $y^{(1)}(0) = -0 \cdot 1 - 1 = -1$
  – Taylor series now say that:

$$y(0 + h) \approx y(0) + y^{(1)}(0)h$$
$$= \quad 1 \quad + (-1)h$$

    - Thus, $y(0.1) \approx 0.9$
  - If $t = 0.1$ and $y(0.1) = 0.9$, then $y^{(1)}(0) = -0.1 \cdot 0.9 - 1 = -1.09$
    - Thus $y(0.2) = y(0.1) + y^{(1)}(0.1) = 0.9 + (-1.09)\ 0.1 = 0.791$

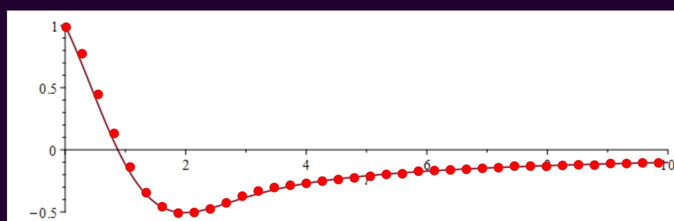
Approximate solutions to IVPs

- In this course,
  we will approximate the solution at specific points:

  $$(t_0, y(t_0)), (t_1, y_1), (t_2, y_2), (t_3, y_3), \ldots$$

  – Thus, $y(t_k) \approx y_k$

This is the initial condition



Approximating at intermediate values of *t*

- Suppose we want to approximate the solution at some point

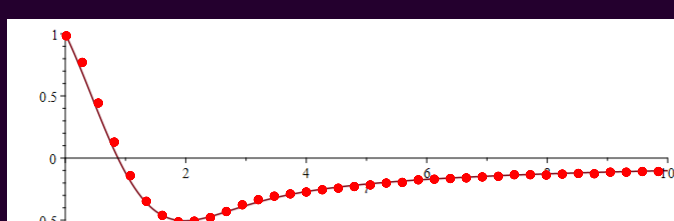
  $$t_{k-1} < t < t_k$$

  – Do we find the interpolating linear polynomial between

  $$(t_{k-1}, y_{k-1}) \text{ and } (t_k, y_k)?$$

  – Do we find the interpolating cubic polynomial between

  $$(t_{k-2}, y_{k-2}), (t_{k-1}, y_{k-1}), (t_k, y_k) \text{ and } (t_{k+1}, y_{k+1})?$$



Approximating solutions to initial-value problems

Interpolating cubic polynomials

- Let's implement this function
  - We assume $t_k - t_{k-1} = h$

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-2} \\ y_{k-1} \\ y_k \\ y_{k+1} \end{pmatrix}$$

```
double ivp_interp_4pt( double t,
                       double ts[4],
                       double ys[4] ) {
    double delta{ (t - ts[1])/(ts[2] - ts[1]) };
    assert( (0.0 <= delta) && (delta <= 1.0) );

    return (
        (
            (
                (ys[3] - ys[0])/6.0 + (ys[1] - ys[2])/2.0
            )*delta + ((ys[0] + ys[2])/2.0 - ys[1])
        )*delta + (-ys[3]/6.0 + ys[2] - ys[1]/2.0 - ys[0]/3.0)
    )*delta + ys[1];
}
```

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Approximating solutions to initial-value problems

Splines

- Recall that $y^{(1)}(t) = f(t, y(t))$, so

$$y^{(1)}(t_{k-1}) = f(t_{k-1}, y_{k-1}) \text{ and } y^{(1)}(t_k) = f(t_k, y_k)$$

  - Can we find a cubic polynomial p that satisfies:

$$p(t_{k-1}) = y_{k-1}$$
$$p^{(1)}(t_{k-1}) = f(t_{k-1}, y_{k-1})$$
$$p(t_k) = y_k$$
$$p^{(1)}(t_k) = f(t_k, y_k)$$

$$p(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$
$$p^{(1)}(t) = 3a_3t^2 + 2a_2t + a_1$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf(t_{k-1}, y_{k-1}) \\ y_k \\ hf(t_k, y_k) \end{pmatrix}$$

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Splines

- This is now fun:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf\left(t_{k-1}, y_{k-1}\right) \\ y_k \\ hf\left(t_k, y_k\right) \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf\left(t_{k-1}, y_{k-1}\right) \\ y_k - y_{k-1} - hf\left(t_{k-1}, y_{k-1}\right) \\ h\left(f\left(t_k, y_k\right) - f\left(t_{k-1}, y_{k-1}\right)\right) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf\left(t_{k-1}, y_{k-1}\right) \\ y_k - y_{k-1} - hf\left(t_{k-1}, y_{k-1}\right) \\ h\left(f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)\right) + 2\left(y_{k-1} - y_k\right) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf\left(t_{k-1}, y_{k-1}\right) \\ 3\left(y_k - y_{k-1}\right) - h\left(2f\left(t_{k-1}, y_{k-1}\right) + f\left(t_k, y_k\right)\right) \\ h\left(f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)\right) + 2\left(y_{k-1} - y_k\right) \end{pmatrix}$$

Splines

- Let's implement this function:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{k-1} \\ hf\left(t_{k-1}, y_{k-1}\right) \\ 3\left(y_k - y_{k-1}\right) - h\left(2f\left(t_{k-1}, y_{k-1}\right) + f\left(t_k, y_k\right)\right) \\ h\left(f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)\right) + 2\left(y_{k-1} - y_k\right) \end{pmatrix}$$

```
double ivp_spline_2pt( double t,     double  ts[2],
                       double ys[2], double dys[2] ) {
    double     h{ ts[1] - ts[0] };
    double delta{ (t - ts[0])/h };
    assert( (0.0 <= delta) && (delta <= 1.0) );

    return (
        (
            (
                h*(dys[0] + dys[1]) + 2.0*(ys[0] - ys[1])
            )*delta - (h*(2.0*dys[0] + dys[1]) + 3.0*(ys[0] - ys[1]))
        )*delta + h*dys[0]
    )*delta + ys[0];
}
```

Approximating at intermediate values of *t*

- Which is better?
  - We will take an IVP to which we know the solution and find:
    1. The linear polynomial interpolating
       (0.20, *y*(0.20)), (0.25, *y*(0.25))
    2. The cubic polynomial interpolating
       (0.15, *y*(0.15)), (0.20, *y*(0.20)), (0.25, *y*(0.25)), (0.30, *y*(0.30))
    3. The cubic spline
       (0.20, *y*(0.20)), (0.25, *y*(0.25))
  - We will then evaluate the actual solution and these approximations at the point $t = 0.2353243$

Approximating at intermediate values of *t*

- First, let's start the 1st-order IVP:

$$y^{(1)}(t) = -y(t) \qquad y(t) = e^{-t}$$
$$y(0) = 1$$

  - Here, *y*(0.2353243) = 0.7903145090700692

Linear interpolating polynomial:	0.7905207882879153
	0.0002062
Cubic interpolating polynomial:	0.7903144140636057
	0.00000009501
Cubic spline:	0.7903144140636057
	0.000000008924

Approximating at intermediate values of *t*

- Next, let's consider :

$$y^{(1)}(t) = \left(t - y(t) + 1\right)\left(y(t) - t\right) \qquad y(t) = t + \frac{1}{2} + \frac{1}{2}\sqrt{3}\tan\left(\frac{\pi - 3\sqrt{3}t}{6}\right)$$

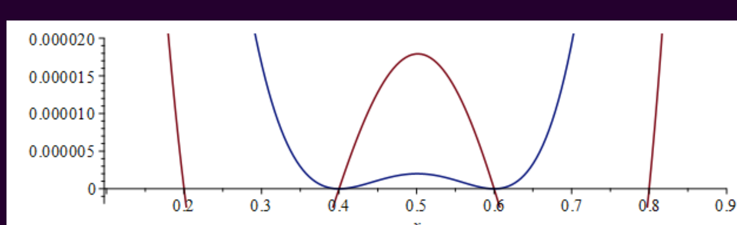
$$y(0) = 1$$

  – Here, $y(0.2353243) = 1.022125607413852$

Linear interpolating polynomial:	1.022252377336976
	0.0001268
Cubic interpolating polynomial:	1.022125194141359
	0.0000004133
Cubic spline:	1.022125568692043
	0.00000003872

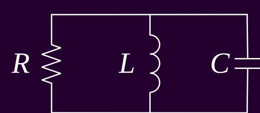
Approximating at intermediate values of *t*

- Also, we can do this with any continuous and differentiable function:
  – Given the sine function, here we see the error of:
    - A cubic polynomial interpolating the values 0.2, 0.4, 0.6, 0.8
    - A cubic spline matching the values and derivatives at 0.4 and 0.6
  – The error of the spline is smaller by a factor of 10



Our approach

- We will begin by approximating the solution to a 1st-order IVP
  - The techniques used here will trivially generalize to allow us to:
    - A system of n coupled 1st-order IVPs

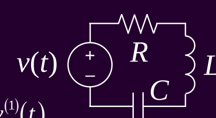


$$v^{(1)}(t) = -\frac{v(t)}{RC} - \frac{i_L(t)}{C}$$

$$i_L^{(1)}(t) = \frac{v(t)}{L}$$

    - An n^{th}-order IVP

$$\theta^{(2)}(t) = -\frac{g}{L}\sin(\theta(t))$$



$$i^{(2)}(t) + \frac{R}{L}i^{(1)}(t) + \frac{1}{CL}i(t) = \frac{1}{L}v^{(1)}(t)$$

    - A system of higher-order coupled IVPs

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Looking ahead

- To approximate a solution to a 1st-order IVP, we will look at:
  - Euler's method
  - Heun's method
  - 4th-order Runge Kutta
  - Adaptive Euler-Heun
  - Dormand-Prince method
  - Stiff ODEs and backward Euler
- We will then generalize these algorithms to approximate the solution to a system of 1st-order coupled IVPs
- We will use such an approach to approximate the solution to an n^{th}-order IVP
- We will then see it is trivial to approximate the solution to a system of higher-order IVPs

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Summary

- Following this topic, you now
  - Understand the various types of initial-value problems
  - Are aware of the approach we will use
  - Know about splines as opposed to interpolating polynomials
  - Are aware that we will approximate solutions to:
    - 1^{st}-order IVPs
    - Systems of 1^{st}-order IVPs
    - Higher-order IVPs
    - Systems of higher-order IVPs

References

[1] https://en.wikipedia.org/wiki/Initial_value_problem

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.
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